Section 4.9

Antiderivatives

If differentiation is the process of finding a derivative f' of the function f, then antidifferentiation is the reverse process:

Definition 4.9.1. The function F is an *antiderivative* of f if $F'(x) = f(x)$.

Our goal in this section is to get as much mileage out of reversing our derivative rules as possible in order to find antiderivatives. Another way of thinking about antiderivatives, $f(x)$ is the antiderivative of $f'(x)$.

Example 4.9.2. If $f(x) = 1$, find functions $F(x)$ such that $F'(x) = f(x)$.

Theorem 4.2.6. If two functions have the same derivative $f'(x) = g'(x)$ for all x in an open interval I, then $f(x) - g(x) = C$ on I; that is, f and g differ by a constant.

Example 4.9.3. Find all antiderivatives for

1. $f(x) = x^2$

2. $g(x) = x^3$

3. $h(x) = 6x^5$

Definition 4.9.4. The symbol which means find all antiderivatives of f is called the *indefinite integral*:

$$\int f(x)\,dx$$

Theorem 4.9.5 (Power Rule for Indefinite Integrals)**.**

$$\int x^n\,dx =$$

where n and C is an arbitrary constant.

Theorem 4.9.6 (Constant Multiple and Sum Rules)**.**

- $\int cf(x)\,dx = c\int f(x)\,dx$
- $\int \big(f(x)+g(x)\big)\,dx = \int f(x)\,dx + \int g(x)\,dx$

Example 4.9.7.

1. $\int \left(3x^2 - \sqrt{x}\right)dx$

2. $\displaystyle\int (z^2 - 3)(z + 1)\, dz$

3. $\displaystyle\int \frac{x^2 + 2}{3x^2} dx$

Indefinite Trig Integrals

$$\frac{d}{dx}\big(\sin x\big) = \cos x \quad \Rightarrow \quad \int \cos x\, dx = \sin x + C$$

$$\frac{d}{dx}\big(\cos x\big) = -\sin x \quad \Rightarrow \quad \int \sin x\, dx = -\cos x + C$$

$$\frac{d}{dx}\big(\tan x\big) = \sec^2 x \quad \Rightarrow \quad \int \sec^2 x\, dx = \tan x + C$$

$$\frac{d}{dx}\big(\sec x\big) = \sec x \tan x \quad \Rightarrow \quad \int \sec x \tan x\, dx = \sec x + C$$

Example 4.9.8. Compute the following indefinite integrals:

1. $\int (2\sin x - \csc^2 x + 3x)\, dx$

2. $\int \left(1 + \tan^2 x\right) dx$

More Antiderivatives

Example 4.9.9. Compute the following indefinite integrals:

1. $\int \frac{1}{x} dx$

2. $\displaystyle\int e^x\,dx$

3. $\displaystyle\int \frac{1}{1+x^2}dx$

$$\frac{d}{dx}\big(e^x\big) = e^x \qquad \Rightarrow \qquad \int e^x\,dx = e^x + C$$

$$\frac{d}{dx}\big(\ln|x|\big) = \frac{1}{x} \qquad \Rightarrow \qquad \int \frac{1}{x}dx = \ln|x| + C$$

$$\frac{d}{dx}\big(\arcsin x\big) = \frac{1}{\sqrt{1-x^2}} \qquad \Rightarrow \qquad \int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C$$

$$\frac{d}{dx}\big(\arctan x\big) = \frac{1}{1+x^2} \qquad \Rightarrow \qquad \int \frac{1}{1+x^2}dx = \arctan x + C$$

$$\frac{d}{dx}\big(\operatorname{arcsec} x\big) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \Rightarrow \qquad \int \frac{1}{|x|\sqrt{x^2-1}}dx = \operatorname{arcsec} x + C$$

Initial Value Problems

We want to find a specific antiderivative $F(x)$ such that $F(a) = b$ for given a and b:

Example 4.9.10. Find $h(t)$ if $h'(t) = 1 + 6\sin t$ and $h(\frac{\pi}{3}) = -3$

Example 4.9.11. Find $y = f(x)$ if $\dfrac{dy}{dx} = 10e^x + 70$ and $p(0) = 100$.